New Exact Solutions for Two Nonlinear Equations

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Nonlinear partial differential equations are widely used to describe complex phenomena in various fields of science, for example the Korteweg-de Vries-Kuramoto-Sivashinsky equation (KdV-KS equation) and the Ablowitz-Kaup-Newell-Segur shallow water wave equation (AKNS-SWW equation). To our knowledge the exact solutions for the first equation were still not obtained and the obtained exact solutions for the second were just N-soliton solutions. In this paper we present kinds of new exact solutions by using the extended tanh-function method.

Key words: KdV-KS Equation; AKNS-SWW Equation; Extended tanh-Function Method; Exact Solution.

1. Introduction

1.1. The Proposed Method

The investigation of the travelling wave solutions of nonlinear equations play an important role in the study of nonlinear wave phenomena. The wave phenomena observed in fluid dynamics, plasma and elastic media are often modelled by the bell-type sech solutions and the kink-type tanh solutions. There has been significant progress in the development of the methods to solve the problem we mentioned, such as inverse scattering method, Darboux transformation [1 – 7], Hirota bilinear method [8 – 10], algebro-geometric method [11 – 14] and tanh method [15]. Among them, the tanh method is considered to be the most effective and direct method for solving nonlinear equations. Recently, much research work has been concentrated on the various extensions and applications of the tanh method [16 – 20]. Our approach stems mainly from Fan and Zhang [21, 22] and Chen et al. [21 – 23]. The difference between our method and theirs is that we adopt a different extension. For given a nonlinear equation

$$H(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{1}$$

the main steps of our proposed are given as follows.

Step 1. By using the wave transformation $u(x,t) = U(\xi)$, $\xi = x + ct$, we reduce (1) into an ordinary differential equation (ODE):

$$H(U, U', U'', \cdots) = 0. \tag{2}$$

Step 2. We introduce a new variable $\omega = \omega(\xi)$ which is a solution of the Riccati equation

$$\omega' = \frac{\mathrm{d}\omega}{\mathrm{d}\xi} = R + \omega^2, \tag{3}$$

where R is a constant to be determined later. Then the derivatives with respect to the variable ξ become the derivatives with respect to the variable ω as

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \to (R+\omega^2)\frac{\mathrm{d}}{\mathrm{d}\omega},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} \to 2\omega(R+\omega^2)\frac{\mathrm{d}}{\mathrm{d}\omega} + (R+\omega^2)\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}, \cdots .$$

Step 3. By virtue of the variable ω, we expend the solution of (4) as

$$u(x,t) = U(\xi) = a_0 + \sum_{i=1}^{n} a_i\omega^i + c_i\omega^{i-1}\sqrt{R+\omega^2}, \tag{4}$$

where a_i, c_i $(i = 1, 2, \cdots, n)$ are also constants to be determined later. This expansion is different from Fan's and Chen's. Fan's expansion is

$$u(x,t) = U(\xi) = a_0 + \sum_{i=1}^{n} a_i\omega^i,$$

Chen's is

$$u(x,t) = U(\xi) = a_0 + \sum_{i=1}^{n} a_i\omega^i + b_i\omega^{-i} + c_i\omega^{i-1}\sqrt{R+\omega^2} + \mathrm{d}_i\frac{\sqrt{R+\omega^2}}{\omega^j}.$$

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Then, balancing the highest derivative term with the nonlinear terms in (2) will give an equation about positive integers *n*, from which the possible value of *n* can be obtained. The value leads to the series expansions of the exact solutions for (1).

Step 4. Substituting the (4) into (2) and setting the coefficients of all powers of ω^i and $\omega^i\sqrt{R+\omega^2}$ to zero, we will get a system of algebraic equations, from which the constants R, c, a_i, c_i $(i = 0, 1, \cdots, n)$ can be found explicitly making use of *Mathematica*.

Step 5. The Riccati equation (3) has the following general solutions:

(i) If $R < 0$,

$$\omega(\xi) = -\sqrt{-R}\tanh(\sqrt{-R}\xi), \tag{5}$$

$$\omega(\xi) = -\sqrt{-R}\coth(\sqrt{-R}\xi). \tag{6}$$

(ii) If $R = 0$,

$$\omega(\xi) = -\frac{1}{\xi}. \tag{7}$$

(iii) If $R > 0$,

$$\omega(\xi) = \sqrt{R}\tan(\sqrt{R}\xi), \tag{8}$$

$$\omega(\xi) = -\sqrt{R}\cot(\sqrt{R}\xi). \tag{9}$$

Substituting the values R, c, a_i, c_i $(i = 0, 1, \cdots, n)$ obtained in Step 4 into (4) – (9),the travelling wave solutions of (3) are obtained.

1.2. The Two Nonlinear Equations

In this paper, we obtain exact solutions for the Korteweg-de Vries-Kuramoto-Sivashinsky (KdV-KS) equation and the Ablowitz-Kaup-Newell-Segur shallow water wave (AKNS-SWW) equation through the proposed method. The KdV-KS equation [24, 25]

$$u_t + \delta u_{xxx} + \beta(u_{xx} + u_{xxxx}) + (u_x)^2 = 0, \tag{10}$$

where $u = u(x,t)$ is a real-valued function and δ, β are constants, was derived independently by Sivashinsky [26] and Kuramoto and Tsuzuki [27]. The KdV-KS equation (10) is a model for amplitude and phase expansion of pattern formations in different physical situations, for example in the theory of a flame propagation in turbulent flows of gaseous combustible mixtures [26], and in the theory of turbulence of wave fronts in reaction-diffusion systems [27]. The Cauchy problem and the global well-posedness for (10) were researched by Bona et al. [28]. The AKNS-SWW equation [29]

$$u_t + u_x + 4uu_t + 2u_x\partial_x^{-1}u_t - u_{xxt} = 0 \tag{11}$$

is one of the well-known shallow water wave equations. Its complete integrability and solvability by the inverse scattering method have been proved by Ablowitz et al. [29], and N-soliton solutions have been given by Hirota and Satsuma [30] via a bilinear method. A catalogue of classical and non-classical symmetry reductions and a Painlevé analysis for the AKNS-SWW equation are given by Clarkson and Mansfield [31]. The present paper is motivated by the desire to find kinds of more and new exact solutions for these two equations by using the extended tanh-function method.

2. Exact Solution for the KdV-KS Equation

In this section, we consider the KdV-KS equation (10). Using the transformation $u(x,t) = U(\xi), \xi = x + ct$, from (10) we get an ordinary differential equation:

$$\begin{aligned} &cU'(\xi) + \delta U'''(\xi) + \beta(U''(\xi) \\ &\quad + U^{(4)}(\xi)) + (U'(\xi))^2 = 0. \end{aligned} \tag{12}$$

According to the method in Section 1, by balancing the highest-order derivative term with the nonlinear term in (12), we get the equation $n + 4 = 2(n + 1)$; so we know $n = 2$. Therefore, based on (4), we expand the solution of (10) as

$$\begin{aligned} u(x,t) = U(\xi) &= a_0 + a_1\omega + a_2\omega^2 \\ &\quad + c_1\sqrt{R+\omega^2} + c_2\omega\sqrt{R+\omega^2}, \end{aligned} \tag{13}$$

where ω satisfies (3). Substituting (13) into (12) and using *Mathematica*, we get an equation about ω^i and $\omega^i\sqrt{R+\omega^2}$. Setting the coefficients of all powers of ω^i and $\omega^i\sqrt{R+\omega^2}$ to zero, we obtain a system of algebraic equations:

$$\begin{aligned} &cRa_1 + 2R^2\delta a_1 + R^2a_1^2 + 2R^2\beta a_2 \\ &\quad + 16R^3\beta a_2 + R^3c_2^2 = 0, \end{aligned}$$

$$\begin{aligned} &2R\beta a_1 + 16R^2\beta a_1 + 2cRa_2 + 16R^2\delta a_2 \\ &\quad + 4R^2a_1a_2 + 2R^2c_1c_2 = 0, \end{aligned}$$

$$\begin{aligned} &ca_1 + 8R\delta a_1 + 2Ra_1^2 + 8R\beta a_2 + 136R^2\beta a_2 \\ &\quad + 4R^2a_2^2 + Rc_1^2 + 5R^2c_2^2 = 0, \end{aligned}$$

$$2\beta a_1+40R\beta a_1+2ca_2+40R\delta a_2 +8Ra_1a_2+6Rc_1c_2=0,$$
$$6\delta a_1+a_1^2+6\beta a_2+240R\beta a_2 +8Ra_2^2+c_1^2+8Rc_2^2=0,$$
$$24\beta a_1+24\delta a_2+4a_1a_2+4c_1c_2=0,$$
$$120\beta a_2+4a_2^2+4c_2^2=0,$$
$$R\beta c_1+5R^2\beta c_1+cRc_2+5R^2\delta c_2+2R^2a_1c_2=0,$$
$$cc_1+5R\delta c_1+2Ra_1c_1+5R\beta c_2 +61R^2\beta c_2+4R^2a_2c_2=0,$$
$$2\beta c_1+28R\beta c_1+4Ra_2c_1+2cc_2 +28R\delta c_2+6Ra_1c_2=0,$$
$$6\delta c_1+2a_1c_1+6\beta c_2+180R\beta c_2+12Ra_2c_2=0,$$
$$24\beta c_1+4a_2c_1+24\delta c_2+4a_1c_2=0,$$
$$120\beta c_2+8a_2c_2=0.$$

With the aid of *Mathematica*, we obtain the solutions of the algebraic system as follows:

$$R=-1,\quad c=-\frac{3\delta}{2},\quad a_1=-\frac{15\delta}{4},\quad a_2=-\frac{15\delta}{4},\quad c_1=\pm\frac{15\delta}{4},\quad c_2=\pm\frac{15\delta}{4},\quad \beta=\frac{\delta}{4};$$
$$R=-\frac{1}{4},\quad c=-\frac{3\delta}{2},\quad a_1=-\frac{15\delta}{2},\quad a_2=\mp\frac{15\delta}{2},\quad c_1=0,\quad c_2=0,\quad \beta=\pm\frac{\delta}{4};$$
$$R=1,\quad c=\delta,\quad a_1=-\frac{15\delta}{4},\quad a_2=-\frac{15\delta}{4},\quad c_1=\pm\frac{15\delta}{4},\quad c_2=\pm\frac{15\delta}{4},\quad \beta=\frac{\delta}{4};$$
$$R=\frac{1}{4},\quad c=\delta,\quad a_1=-\frac{15\delta}{2},\quad a_2=\mp\frac{15\delta}{2},\quad c_1=0,\quad c_2=0,\quad \beta=\pm\frac{\delta}{4};$$
$$R=-1,\quad c=-\frac{3\delta}{2},\quad a_1=-\frac{15\delta}{4},\quad a_2=\frac{15\delta}{4},\quad c_1=\pm\frac{15\delta}{4},\quad c_2=\mp\frac{15\delta}{4},\quad \beta=-\frac{\delta}{4};$$
$$R=1,\quad c=\delta,\quad a_1=-\frac{15\delta}{4},\quad a_2=\frac{15\delta}{4},\quad c_1=\pm\frac{15\delta}{4},\quad c_2=\mp\frac{15\delta}{4},\quad \beta=-\frac{\delta}{4}.$$

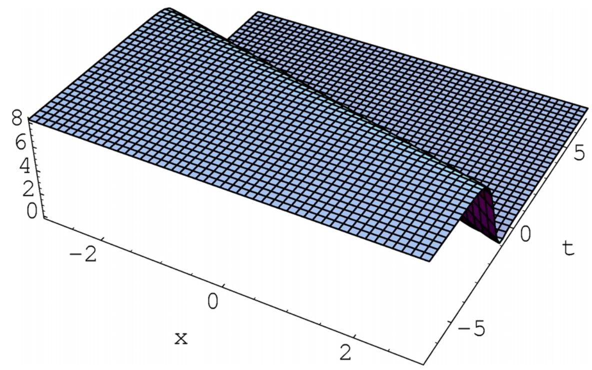

Fig. 1. The kink-type solution u_1 with $\delta=-1, a_0=0.75$.

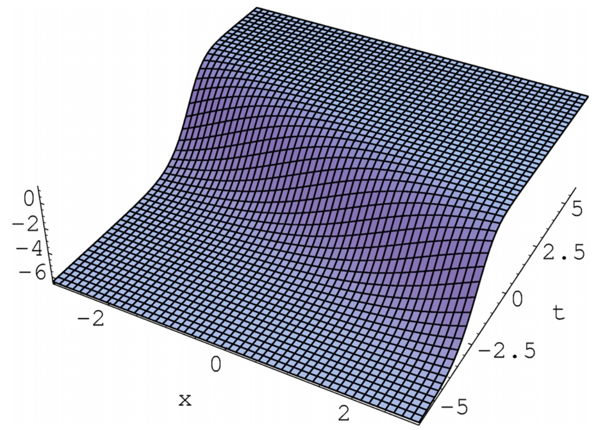

Fig. 2. The kink-type solution u_2 with $\delta=-1, a_0=0.75$.

According to the result, $R\neq 0$ and, only under the condition $\beta=\pm\frac{\delta}{4}$, the algebraic system has solutions which lead valuable exact solutions for (10).

Case 1. $R=-1$, according to (5), we obtain a kink-type solution (Fig. 1):

$$u_1=a_0+\frac{15\delta}{4}\tanh\left(x-\frac{3\delta}{2}t\right)-\frac{15\delta}{4}\tanh^2\left(x-\frac{3\delta}{2}t\right) \pm\frac{15\delta}{4}\mathrm{i}\,\mathrm{sech}\left(x-\frac{3\delta}{2}t\right) \mp\frac{15\delta}{4}\mathrm{i}\tanh\left(x-\frac{3\delta}{2}t\right)\mathrm{sech}\left(x-\frac{3\delta}{2}t\right).$$

Case 2. $R=-\frac{1}{4}$, according to (5), we obtain a kink-type solution (Fig. 2):

$$u_2=a_0+\frac{15\delta}{4}\tanh\left(\frac{x}{2}-\frac{3\delta}{4}t\right)-\frac{15\delta}{8}\tanh^2\left(\frac{x}{2}-\frac{3\delta}{4}t\right).$$

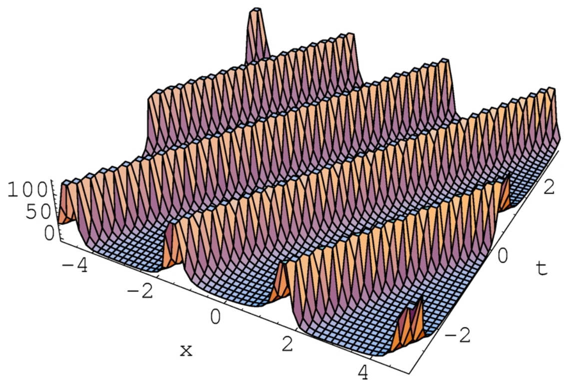

Fig. 3. The periodic wave solution u_3 with $\delta = -1, a_0 = 1$.

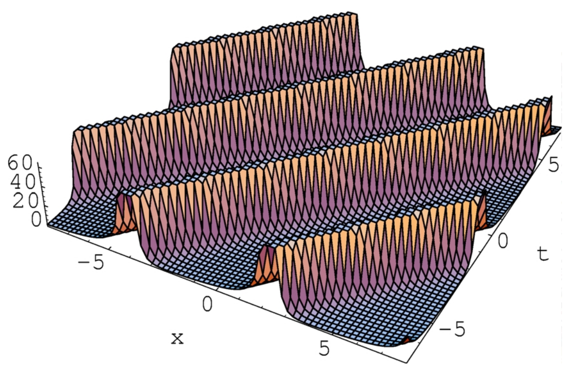

Fig. 4. The periodic wave solution u_4 with $\delta = -1, a_0 = 1$.

Case 3. $R = 1$, according to (6), we obtain a periodic wave solution (Fig. 3):

$$u_3 = a_0 - \frac{15\delta}{4}\tan(x+\delta t) - \frac{15\delta}{4}\tan^2(x+\delta t) \pm \frac{15\delta}{4}\sec(x+\delta t) \pm \frac{15\delta}{4}\tan(x+\delta t)\sec(x+\delta t),$$

Case 4. $R = \frac{1}{4}$, according to (6), we obtain a periodic wave solution (Fig. 4):

$$u_4 = a_0 - \frac{15\delta}{4}\tan(\frac{x}{2}+\frac{\delta}{2}t) - \frac{15\delta}{8}\tan^2(\frac{x}{2}+\frac{\delta}{2}t).$$

These four cases satisfy the condition $\beta = \frac{\delta}{4}$.

We draw some plots for some formal solutions, so that we can learn the properties of these solutions (Figs. 1–4).

Remark 1. Since cot- and coth-type solutions appear in pairs with tan- and tanh-type solutions, respectively, they are omitted in this section.

Remark 2. When $\beta = -\frac{\delta}{4}$, the solutions we get are similar with the ones we have obtained, so they are omitted.

3. Exact Solution for the AKNS-SWW Equation

In this section, we consider the AKNS-SWW equation (11). Just the same as in Section 2, we use the transformation $u(x,t) = U(\xi), \xi = x + ct$. According to (11) we get an ordinary differential equation

$$(c+1)U'(\xi) + 6cU(\xi)U'(\xi) - cU'''(\xi) = 0. \quad (14)$$

According to the method in Section 1, by balancing the highest-order derivative term with the nonlinear term in (14), we get the equation $n+3 = 2n+1$; so we know $n = 2$. Based on (4), we can also expand the solution of (11) as (13), where ω satisfies (3). Substituting (13) into (14) and using *Mathematica*, we get an equation about ω^i and $\omega^i\sqrt{R+\omega^2}$. Setting the coefficients of all powers of ω^i and $\omega^i\sqrt{R+\omega^2}$ to zero, we obtain a system of algebraic equations:

$$\begin{aligned}
&Ra_1 + cRa_1 - 2cR^2a_1 + 6cRa_0a_1 + 6cR^2c_1c_2 = 0,\\
&6cRa_1^2 + 2Ra_2 + 2cRa_2 - 16cR^2a_2 + 12cRa_0a_2 + 6cRc_1^2 + 6cR^2c_2^2 = 0,\\
&a_1 + ca_1 - 8cRa_1 + 6ca_0a_1 + 18cRa_1a_2 + 24cRc_1c_2 = 0,\\
&6ca_1^2 + 2a_2 + 2ca_2 - 40cRa_2 + 12ca_0a_2 + 12cRa_2^2 + 6cc_1^2 + 18cRc_2^2 = 0,\\
&-6ca_1 + 18ca_1a_2 + 18cc_1c_2 = 0,\\
&-24ca_2 + 12ca_2^2 + 12cc_2^2 = 0,\\
&6cRa_1c_1 + Rc_2 + cRc_2 - 5cR^2c_2 + 6cRa_0c_2 = 0,\\
&c_1 + cc_1 - 5cRc_1 + 6ca_0c_1 + 12cRa_2c_1 + 12cRa_1c_2 = 0,\\
&12ca_1c_1 + 2c_2 + 2cc_2 - 28cRc_2 + 12ca_0c_2 + 18cRa_2c_2 = 0,\\
&-6cc_1 + 18ca_2c_1 + 18ca_1c_2 = 0,\\
&-24cc_2 + 24ca_2c_2 = 0.
\end{aligned}$$

With the aid of *Mathematica*, we solve the algebraic system, and obtain c, a_1, a_2, c_1, c_2:

$$c = \frac{1}{-1+5R-6a_0}, \quad a_1 = 0, \quad a_2 = 1, \quad c_1 = 0, \quad c_2 = \pm 1;$$

$$c = \frac{1}{-1+8R-6a_0}, \quad a_1 = 0, \quad a_2 = 2, \quad c_1 = 0, \quad c_2 = 0,$$

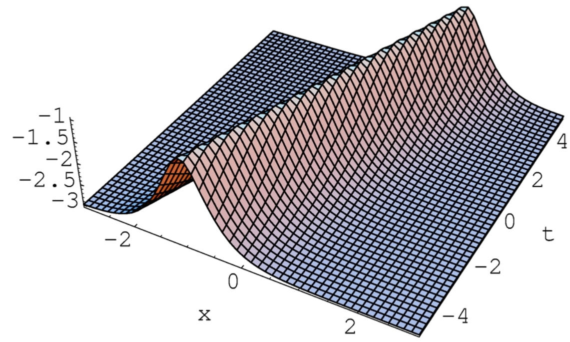

Fig. 5. The bell-type solution u_1 with $a_0 = 1, R = -2$.

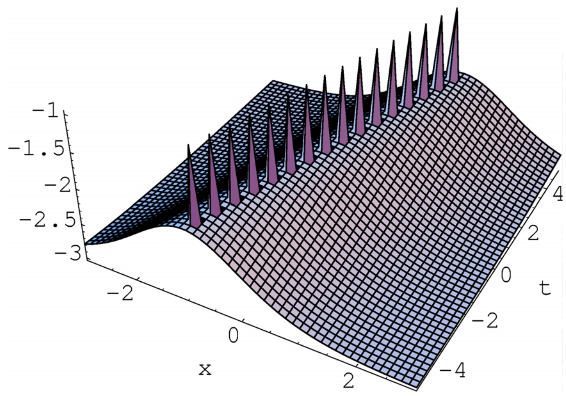

Fig. 6. The singular solution u_2 with $a_0 = 1, R = -2$.

where R is an arbitrary constant. Substituting them in (13), some kinds of exact solutions for (11) are shown.

Case 1. $R < 0$, according to (5) and (6), we obtain two bell-type solutions, a singular solution and a rational solution (Figs. 5 – 8):

$$u_1 = a_0 - R\tanh^2\left[\sqrt{-R}\left(x + \frac{1}{-1+5R-6a_0}t\right)\right]$$
$$\pm \mathrm{i}R\tanh\left[\sqrt{-R}\left(x + \frac{1}{-1+5R-6a_0}t\right)\right]$$
$$\cdot \operatorname{sech}\left[\sqrt{-R}\left(x + \frac{1}{-1+5R-6a_0}t\right)\right],$$

$$u_2 = a_0 - R\coth^2\left[\sqrt{-R}\left(x + \frac{1}{-1+5R-6a_0}t\right)\right]$$
$$\pm R\coth\left[\sqrt{-R}\left(x + \frac{1}{-1+5R-6a_0}t\right)\right]$$
$$\cdot \operatorname{csch}\left[\sqrt{-R}\left(x + \frac{1}{-1+5R-6a_0}t\right)\right],$$

$$u_3 = a_0 - 2R\tanh^2\left[\sqrt{-R}\left(x + \frac{1}{-1+8R-6a_0}t\right)\right],$$

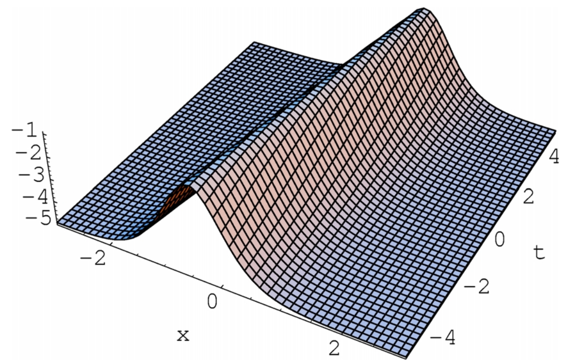

Fig. 7. The bell-type solution u_3 with $a_0 = 1, R = -2$.

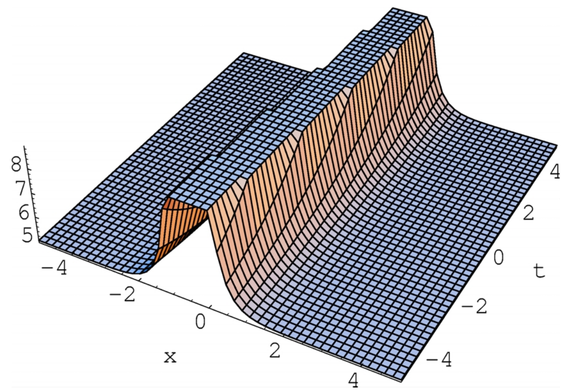

Fig. 8. The rational solution u_4 with $a_0 = 1, R = -2$.

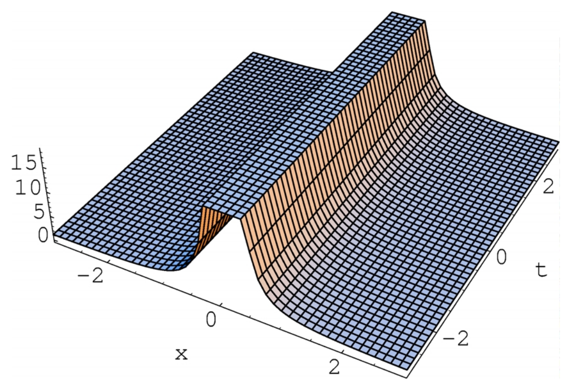

Fig. 9. The rational solution u_5 with $a_0 = 1, R = 0$.

$$u_4 = a_0 - 2R\coth^2\left[\sqrt{-R}\left(x + \frac{1}{-1+8R-6a_0}t\right)\right].$$

Case 2. $R = 0$, according to (7), we obtain a rational solution (Fig. 9):

$$u_5 = a_0 + \frac{2}{(x - \frac{1}{1+6a_0}t)^2}.$$

The plots for the solutions are also given, so that the properties of these solutions are shown (Figs. 5 – 9).

Remark 1. Since tan- and cot-type solutions (when $R > 0$) appear in pairs with tanh- and coth-type solutions, respectively, they are omitted in this section.

Remark 2. Although we can not find the N-soliton solutions for (11), which Hirota obtained. We get not only solitary solutions, but also more other kinds of solutions.

4. Conclusions

In this paper, making use of the extended tanh-function method, we successfully obtain the exact travelling wave solutions for the KdV-KS equation and the AKNS-SWW equation. What's more, the properties of the solutions of the two equations have been shown clearly by means of their figures.

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